

YITP-98-82

Dec. 1998

Apr. 1999 (revised & title changed)

Averaging from a global point of view

MASAYUKI TANIMOTO*

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan.

ABSTRACT

We study the averaging problem from a point of view of variation of spatial volume V . We show that in the space of spherically symmetric dust solutions which are regular on the spatial manifold S^3 the variation δV vanishes at the Friedmann-Lemaître-Robertson-Walker (FLRW) solution in an appropriate sense, which supports the validity of the FLRW solution as the averaged solution. We also present the second variation $\delta^2 V$, giving the leading effect of the deviation from the FLRW solution.

*JSPS Research Fellow. Electronic mail: tanimoto@yukawa.kyoto-u.ac.jp

I. INTRODUCTION

The standard cosmology is based on the assumption that our Universe is homogeneous and isotropic. However, the present our Universe is not homogeneous but has clumpy structures like stars, galaxies, clusters of galaxies, and superclusters. The recent observation [1] of the highly isotropic cosmic microwave background radiation is usually regarded as an evidence of the assumption, but this is the case only up to the stage of decoupling. Nonetheless, one may want to think that the homogeneous and isotropic universe model, known as the Friedmann-Lemaître-Robertson-Walker (FLRW) model, reflects the averaged nature of the true Universe over a scale larger than a supercluster. In particular, we usually expect that the global expansion of the Universe is well approximated by the FLRW model with the energy distribution given by the volume average.

However, this “averaging hypothesis” is never justified in a trivial way [2], since Einstein’s equation is highly nonlinear. In particular, averaging over a (spatial) volume does not commute with the time-evolution in general, so the averaged initial data can develop in time in a quite different way from the true data. One is therefore required to clarify the meaning of the averaging and establish the domain of applicability of it.

One of the earliest work connected to this problem is due to Futamase [3], who built a formalism which gives the back reaction of small scale inhomogeneities to the global expansion along the spirit of Isaacson [4] using the post-Newtonian expansion. However, it is difficult to justify the approximation used in this approach, and the basic equations are still hard to handle. Zalaletdinov [5] proposed a covariant averaging formalism starting from some axioms, but this is so complicated that it seems very difficult to draw useful consequences. Up to now, in spite of efforts [6–9] including the above, the averaging has not been well understood, because of its complexities and conceptual difficulties.

In this paper we study the problem in a quite different way from the conventional ones. We focus on the dynamics of the scale factor $a(\tau)$, which is, if the space is closed [10], equivalent to the dynamics of the total volume $V(\tau)$ in the sense $V(\tau) \propto a^3(\tau)$. We examine to what extent the FLRW solution can be a good model judging from a point of view of the time-development of the total volume $V(\tau)$. Note that when given a space of solutions spanned by some arbitrary functions of space, we can think of $V(\tau)$ as a functional on it. By evaluating the variation of $V(\tau)$ at the FLRW solution we may be able to obtain some information about the quality of the FLRW solution as an “averaged” model. We explicitly do this for the spherically symmetric dust case. As a result, we find that the FLRW solution is a *critical point* for $V(\tau)$ in an appropriate sense, which gives a good support for the validity of the FLRW solution as an averaged model. We also evaluate the second variation of $V(\tau)$, which gives the leading effect of the deviation from the FLRW solution.

The exact solution for spatially spherically symmetric dust spacetimes is known as the

Lemaître-Tolman-Bondi (LTB) solution. We add to this solution the assumption that the spatial manifold be S^3 . We give a description of this subclass of the LTB solution first.

II. THE SPHERICALLY SYMMETRIC DUST SOLUTION ON S^3

The metric is written with the synchronous and comoving coordinates as:

$$ds^2 = d\tau^2 - e^{\lambda(\tau, R)} dR^2 - r^2(\tau, R)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

where $\lambda(\tau, R)$ and $r(\tau, R)$ are the functions to be determined from the Einstein equation. The general solution to this metric is well known as the Lemaître-Tolman-Bondi (LTB) solution (see e.g. [11,12]), which possesses three arbitrary functions $f(R)$, $F(R)$, and $\tau_0(R)$. (Our notation follows Ref. [11], except for the sign of $f(R)$.) With these the function λ is given by $e^\lambda = r'^2/(1 - f(R))$, where the dash stands for the derivative with respect to R . The function r is given in three separate forms, depending upon whether the arbitrary function $f(R)$ is negative, positive, or zero, and each solution possesses the FLRW limit of negative ($k = -1$), positive ($k = 1$), and flat ($k = 0$) constant curvature, respectively. (Such a limit is achieved when $r(\tau, R)$ separates as $r(\tau, R) = \Phi(\tau)\Psi(R)$ [12].) Since our model is spatially S^3 , the spatial manifold does admit a constant curvature limit and it should be positive. So, it is natural to choose the positive sign of $f(R)$ [13], for which the function r is given by

$$r = \frac{F}{2f}(1 - \cos \eta), \quad \tau - \tau_0(R) = \frac{F}{2f^{3/2}}(\eta - \sin \eta). \quad (2)$$

The arbitrary function $\tau_0(R)$ is called the function of “big bang time”, since each Killing orbit ($R = \text{constant}$) degenerates at time $\tau = \tau_0(R)$.

Now we have seen that the solution is parametrized by three “arbitrary” functions, but not all solutions are suitable for the spatial manifold S^3 . The aim of the remaining part of this section is to describe the conditions imposed on the three functions to obtain “regular” solutions on S^3 . (By a regular solution we mean that it is regular on the spatial manifold S^3 *during a finite interval of time* from the big bang.) We, by describing them, show that there does exist sufficiently large set of the regular solutions in the space of all formal solutions. This will be needed to make the formal calculations in the next section realistic.

Let us think of the three-sphere as a sum of two balls, $S^3 \simeq D^3 \cup D^3$. This is similar to the decomposition of two-sphere into two disks, $S^2 \simeq D^2 \cup D^2$, achieved by cutting the two-sphere along the equator. We can understand the spherically symmetric three-sphere by thinking of each ball spherically symmetric. Since each such ball has a symmetry center, the spherically symmetric three-sphere has *two* symmetry centers, which are degenerate Killing

orbits, as well. We label these points as $R = 0$ and $R = \pi$, so the relevant region of the coordinate R is $I \equiv [0, \pi]$. The first condition we should impose is therefore

$$r = 0, \text{ at } R = 0 \text{ and } \pi. \quad (3)$$

The function r should be positive except at the boundaries. Furthermore, we must impose some regularity conditions. An efficient way to see this is to calculate the scalar curvature \mathcal{R} and the curvature scalar polynomial $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$, where \mathcal{R}_{abcd} is the curvature tensor. Using our local solution, we find $\mathcal{R} = \varepsilon$ and $\mathcal{R}_{abcd}\mathcal{R}^{abcd} = 12(F/r^3)^2 - 8(F/r^3)\varepsilon + 3\varepsilon^2$, where $\varepsilon = F'/(r'r^2)$ is the energy density of dust. Since these scalars must be finite, we have

$$\frac{F}{r^3} < \infty, \quad R \in [0, \pi] \quad (4)$$

and

$$\varepsilon = \frac{F'}{r'r^2} < \infty, \quad R \in [0, \pi]. \quad (5)$$

While condition (4) is necessary to avoid the conical singularity possibly generated at $R = 0$ and π , condition (5) is imposed to avoid the well-known shell-crossing singularity [14]. Finally, we have to impose the following coordinate condition, which is necessary for the coordinates to span the spatial manifold S^3 well;

$$e^\lambda = \frac{r'^2}{1-f} > 0, \quad R \in [0, \pi]. \quad (6)$$

Without this condition, we possibly have fictitious solutions.

To find the solution to the boundary conditions, note that there exists the freedom of reparameterizations $R \rightarrow \gamma(R)$. Using this freedom we can fix the leading power of r at the boundaries to the unity, i.e., we can make

$$r \propto \begin{cases} R & (R \rightarrow 0) \\ \pi - R & (R \rightarrow \pi) \end{cases}. \quad (7)$$

Near the big bang singularity ($\eta \rightarrow 0$), from Eq.(2) we have $r \sim (F/(4f))\eta^2$, and $\tau - \tau_0(R) \sim (F/(12f^{3/2}))\eta^3$, so we find $r \sim (9/4)^{1/3}F^{1/3}(\tau - \tau_0(R))^{2/3}$. Taking the condition (7) into account, we have

$$F(R) \propto \begin{cases} R^3 & (R \rightarrow 0) \\ (\pi - R)^3 & (R \rightarrow \pi) \end{cases}. \quad (8)$$

Let σ be a real number, and suppose $\eta \propto R^\sigma$ ($R \rightarrow 0$) for an arbitrary fixed τ . From Eq.(2), we find that if $\sigma < 0$, the function r would oscillate heavily between positive values and zero near the boundary. So, this case is unsuitable for the condition. On the other

hand, if $\sigma \geq 0$, the boundary condition (3) is satisfied and the behavior of $f(R)$ near the boundaries can be determined as

$$f(R) \propto \begin{cases} R^{\alpha_1} & (R \rightarrow 0) \\ (\pi - R)^{\alpha_2} & (R \rightarrow \pi) \end{cases}, \quad (9)$$

where $\alpha_1, \alpha_2 \geq 2$. The condition (4) is now trivially satisfied.

Next, consider the regularity conditions (5) and (6). Note that the function r on I for a fixed τ has at least one extremum (FIG.1), since $r > 0$ on the interior of I and $r = 0$ at the boundaries. From the condition (5), we find that the function F' should vanish where r' vanishes (FIG.2). Since $F(R)$ is independent of τ , r' should also vanish on I independently from τ . Otherwise, the regularity would break instantaneously. A straightforward calculation gives

$$r' = \frac{1}{1 - \cos \eta} \left[\frac{F'}{2f} \mathcal{A}(\eta) + \frac{Ff'}{2f^2} \mathcal{B}(\eta) - f^{1/2} \tau'_0 \sin \eta \right], \quad (10)$$

where $\mathcal{A}(\eta) \equiv (1 - \cos \eta)^2 - \sin \eta(\eta - \sin \eta)$, and $\mathcal{B}(\eta) \equiv -(1 - \cos \eta)^2 + \frac{3}{2} \sin \eta(\eta - \sin \eta)$. Since r' is a homogeneous linear combination of f' , F' , and τ'_0 with distinct coefficients as functions of τ , the only possible points on I for which r' vanish independently from τ are the points where F' , f' and τ'_0 vanish simultaneously. In particular, r' and F' should have the same sign everywhere on I since otherwise the energy density ε would become negative in some regions. We exclude such a case for physical reason. On the other hand, the sign of f' is rather arbitrary, except that $f(R)$ should be maximal and take value 1 where F' vanishes (FIG.3). Taking value 1 is a consequence of the condition (6). The sign of τ'_0 should be opposite to F' or zero (FIG.4). This is a consequence of the condition that r' have the same sign as F' at least for a finite interval of time from the big bang ($\eta = 0$). In fact, the leading powers of $\mathcal{A}(\eta)$, $\mathcal{B}(\eta)$, and $\sin \eta$ in Eq.(10) are, respectively, $\frac{1}{12}\eta^4$, $-\frac{1}{80}\eta^6$, and η , so the term proportional to τ'_0 dominates near the big bang singularity if τ'_0 does not vanish. It is clear that if τ'_0 had the same sign as F' , r' would have the opposite sign when $\eta \rightarrow 0$.

Now, we have spelled out all the conditions for $F(R) \geq 0$, $f(R) \geq 0$, and $\tau_0(R)$. They are Eqs.(8) and (9), that $f(R)$ should take maximum value 1 where F' vanishes, and that the sign of τ'_0 should be opposite to F' or zero. It is nice to keep in mind that the profile of $F(R)$ decides that of r if the solution is regular.

III. VARIATION OF V AND AVERAGING

Given an inhomogeneous solution, in what way can we measure the resemblance between the solution and the FLRW solution? We note that the only dynamical content of the FLRW model is the scale factor $a(\tau)$, which can be regarded as the (cubic root of the) total volume

$V(\tau)$ if the spatial manifold is closed [10]. Since $V(\tau)$ is always well defined for any spatially closed inhomogeneous spacetime [15] there is a large amount of theoretical advantage to utilize it. Note that V can be considered as a functional $V[\phi_1(\mathbf{x}), \dots \phi_n(\mathbf{x})]$ on a space of solutions, where $\phi_1(\mathbf{x}), \dots \phi_n(\mathbf{x})$ are the arbitrary functions of space which span the space of solutions. For example, in the spherically symmetric dust case, $n = 3$ and we can put $(\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \phi_3(\mathbf{x})) \equiv (F(R), f(R), \tau_0(R))$. If the inhomogeneous solution is not too far from the FLRW solution, we can compare the two solutions by evaluating the variation $\delta V(\tau)$ at the FLRW solution. If this function $\delta V(\tau)$ vanishes for all τ , we may say that the two solutions are dynamically close to each other and the “averaged” solution of the inhomogeneous solution corresponds to the FLRW solution.

Below, we apply the above idea to the spherically symmetric dust solution on S^3 . Our space P of solutions are assumed to be spanned by the regular solutions on S^3 , and the variations are taken in it. More specifically, P is a subspace of the larger space P^* which is spanned by all possible configurations of three smooth functions $(F(R), f(R), \tau_0(R))$, and defined by subtracting from P^* the configurations which correspond to irregular solutions. The space P contains the FLRW solution, since this solution is regular.

The total volume for the metric (1) is

$$V(\tau) = 4\pi \int_0^\pi e^{\lambda/2} r^2 dR, \quad (11)$$

where $e^\lambda = r'^2/(1 - f(R))$ and r is given by Eq.(2). For later use, we also define the total energy E :

$$E \equiv \int_{S^3} \varepsilon dV = 4\pi \int_0^\pi \frac{|F'|}{\sqrt{1-f}} dR, \quad (12)$$

which is a conserved quantity. Moreover, for convenience we change the parameterization $(F(R), f(R), \tau_0(R))$ to $(A(R), f(R), \tau_0(R))$ defined by $A(R) \equiv F/(2f^{3/2})$. The standard FLRW limit is achieved in this parameterization when

$$(A(R), f(R), \tau_0(R)) = (a_0, \sin^2 R, \tau_{0c}), \quad (13)$$

where a_0 is a positive constant parameter, and τ_{0c} is another constant parameter. Since we can always choose $\tau_{0c} = 0$ by the coordinate transformation $\tau \rightarrow \tau + \tau_{0c}$, parameter τ_{0c} is redundant as far as the FLRW solution is concerned, but we will find that in our wider context it is useful not to fix this parameter. At the limit (13), V is given by

$$V_0(\tau) = 2\pi^2 a_0^3 (1 - \cos \eta)^3, \quad \tau - \tau_{0c} = a_0(\eta - \sin \eta). \quad (14)$$

We vary the volume V with respect to $A(R)$, $f(R)$, and $\tau_0(R)$, and evaluate at the FLRW solution (13). The formal result is

$$\begin{aligned}
\delta V(\tau) = & 4\pi a_0^2(1 - \cos \eta) \mathcal{A}(\eta) \int_0^\pi (3\delta A + \tan R \delta A') \sin^2 R \, dR \\
& + 2\pi a_0^3(1 - \cos \eta)^3 \int_0^\pi (\tan R \delta f)' \, dR \\
& - 4\pi a_0^2(1 - \cos \eta) \sin \eta \int_0^\pi (3\delta \tau_0 + \tan R \delta \tau_0') \sin^2 R \, dR.
\end{aligned} \tag{15}$$

We may expect that the second term vanishes if we take into account the boundary conditions in the previous section, but we have to check the continuity of the function $\tan R \delta f$ at $R = \pi/2$ to do this. The zeroth variation of $f(R)$ is $\sin^2 R$ and the maximal value should always be 1 as we explained in the previous section. This means that at the neighborhood of $R = \pi/2$, the significant (or “dangerous”) variation of $f(R)$ should always be approximated by $\sin^2(R + a)$, where a is a parameter. Differentiating this with respect to a and putting $a = 0$, we find that δf is approximated by $2 \sin R \cos R \, da$, so that $\tan R \delta f$ is approximated by $2 \sin^2 R \, da$, which is continuous at $R = \pi/2$. We can hence safely omit the term. (In contrast to this, we cannot apply “integrations by parts” to the first and the third terms due to the discontinuities of $\sin^2 R \tan R \delta A$ and the similar term for $\delta \tau_0$.) Thus we have

$$\begin{aligned}
\delta V(\tau) = & 4\pi a_0^2(1 - \cos \eta) \mathcal{A}(\eta) \int_0^\pi (3\delta A + \tan R \delta A') \sin^2 R \, dR \\
& - 4\pi a_0^2(1 - \cos \eta) \sin \eta \int_0^\pi (3\delta \tau_0 + \tan R \delta \tau_0') \sin^2 R \, dR.
\end{aligned} \tag{16}$$

Note that the function η depends only on τ when it concerns the FLRW solution, so the functions of η in the above expressions have been factored out of the integrals. Because of this feature $\delta V(\tau)$ becomes constantly zero, if (and only if) the variation of $A(R)$ and $\tau_0(R)$ are taken so that the integrals in Eq.(16) vanish. This means that the FLRW solution is a critical point for the functional $V(\tau)$, if we restrict the directions of the variation in that way.

The meaning of the particular directions becomes clear if we consider the variation of the total energy E , which coincides with the first term of the rhs of Eq.(16) up to prefactor:

$$\delta E = 8\pi \int_0^\pi (3\delta A + \tan R \delta A') \sin^2 R \, dR. \tag{17}$$

(We have dropped the zero term $12\pi a_0 \int_0^\pi (\tan R \delta f)' \, dR$.) One may notice that the second term of the rhs of Eq.(16) is given similarly by the variation of

$$C \equiv -4\pi \int_0^\pi \frac{|(f^{3/2} \tau_0)'|}{\sqrt{1-f}} \, dR, \tag{18}$$

which has been defined by replacing $A(R)$ by $\tau_0(R)$ (and multiplying the factor $-1/2$) in the definition of E . In fact, we obtain

$$\delta C = -4\pi \int_0^\pi (3\delta \tau_0 + \tan R \delta \tau_0') \sin^2 R \, dR, \tag{19}$$

so that we can write the formula (16) as

$$\delta V(\tau) = a_0^2(1 - \cos \eta) \left(\mathcal{A}(\eta) \frac{\delta E}{2} + \sin \eta \delta C \right). \quad (20)$$

Here, note that the space of solutions P is foliated by the surface $P_{E,C}$ of constant E and C , while the FLRW solution O is a two dimensional subset in P spanned by the two parameters a_0 and τ_{0c} . We write the FLRW solution with fixed values of a_0 and τ_{0c} as $O_{a_0, \tau_{0c}} \in O$. We can easily find that there is a unique FLRW solution $O_{a_0, \tau_{0c}}$ in each $P_{E,C}$, and we can define a map $Av : P \rightarrow O$ by this correspondence:

$$Av : P_{E,C} \rightarrow O_{a_0, \tau_{0c}}. \quad (21)$$

The significance of Eq.(20) is that the FLRW solution $O_{a_0, \tau_{0c}}$ that best matches a given inhomogeneous solution $p \in P$ is given by $Av(p)$, the FLRW solution with the same energy E and the same “ C ”. That is, since in each surface $P_{E,C}$ the FLRW solution $Av(P_{E,C})$ is the critical point for the volume $V(\tau)$, all the inhomogeneous solutions which are sufficiently close to $Av(P_{E,C})$ virtually manifest the same dynamical evolutions of volume as that of $Av(P_{E,C})$. This is our main result. We will call Av the *averaging map*.

The key relation (20) seems very reasonable (though it is not trivial) if we note the following fact about the FLRW solution. That is, if we vary the two FLRW parameters a_0 and τ_{0c} the total volume $V_0(\tau)$ for the FLRW solution will suffer a certain change. We can easily estimate it by differentiating $V_0(\tau)$ with respect to the two parameters. (See Eq.(14).) The result is

$$dV_0(\tau) = 6\pi^2 a_0^2 (1 - \cos \eta) (2\mathcal{A}(\eta) da_0 - \sin \eta d\tau_{0c}). \quad (22)$$

Moreover, using

$$E_0 = 12\pi^2 a_0, \quad C_0 = -6\pi^2 \tau_{0c} \quad (23)$$

for the value of E and C at the FLRW limit (13), we obtain exactly the same formula as Eq.(20):

$$dV_0(\tau) = a_0^2(1 - \cos \eta) \left(\mathcal{A}(\eta) \frac{dE_0}{2} + \sin \eta dC_0 \right). \quad (24)$$

In this sense the functionals E and C are “inhomogeneous generalizations” of the FLRW parameters a_0 and τ_{0c} . Relations (23) also imply

$$Av(P_{E,C}) = O_{a_0 = \frac{E}{12\pi^2}, \tau_{0c} = \frac{-C}{6\pi^2}}. \quad (25)$$

Recall that the parameter τ_{0c} contained in the FLRW solution is merely a kind of gauge freedom, and therefore we could fix it like $\tau_{0c} = 0$. This seems to mean that we can restrict

the space of solution P to the slice $C = 0$ by gauge fixing. This is in fact the case. Note that the coordinate transformation $\tau \rightarrow \tau - c$, where c is a constant, induces the transformation $\tau_0(R) \rightarrow \tau_0(R) + c$. Hence the space of solutions P contains gauge freedom of this type. Since the above transformation for $\tau_0(R)$ shifts the value of C by $4\pi c \int_0^\pi \frac{|f^{3/2}|}{\sqrt{1-f}} dR$, every surface of constant C corresponds to the same set of spacetime solutions, and this shows the claim. Note that the gauge freedom we consider is the freedom of choosing the origin of the time coordinate. The restriction $C = 0$ therefore provides us a natural way of choosing the origin of the time coordinate for every distinct inhomogeneous solutions, especially with non-simultaneous big bangs, viewing from the comparisons of $V(\tau)$.

We should comment on a specialty of the FLRW solution. It was natural to naively expect that the FLRW solution describes the averaged spacetime of an inhomogeneous spacetime that is homogeneous and isotropic over a large scale. We have seen that this is in fact the case for the spherically symmetric case in the sense $\delta V(\tau) = 0$ at the FLRW solution. One might similarly expect that a spacetime that is smooth over a large scale but fluctuates over small scales could be approximated by a smoothed solution, or one might expect that a homogeneous but anisotropic (i.e., Bianchi or Kantowski-Sachs-Nariai type [16]) spacetime solution could be served as a good model for an inhomogeneous spacetime that is homogeneous over a large scale. However, at least for the spherically symmetric case only the FLRW solution is the good averaged or smoothed spacetime model, since there does not exist a point where the function $\eta(\tau, R)$ becomes a function of only time τ in P , except at the FLRW solution, i.e., $\delta V(\tau) = 0$ holds only at the FLRW solution.

The accuracy of the FLRW solution as the averaged one can be estimated from the second variation $\delta^2 V$. That is, we first parameterize the arbitrary functions with one parameter ϵ so that $\epsilon = 0$ corresponds to the FLRW solution, and expand them in power of ϵ : $A(R; \epsilon) = a_0 + \epsilon \delta A(R) + (1/2)\epsilon^2 \delta^2 A(R) + \dots$, and similarly for $f(R; \epsilon)$ and $\tau_0(R; \epsilon)$. Then, we expand $V[A(R; \epsilon), f(R; \epsilon), \tau_0(R; \epsilon)]$ in accordance with these expansions;

$$V = V_0 + \epsilon \delta V + \frac{1}{2} \epsilon^2 \delta^2 V + \dots \quad (26)$$

If the first order term vanishes as in our case, the second order term $\frac{1}{2} \epsilon^2 \delta^2 V$ gives the leading term for the deviation from the FLRW solution, i.e., the accuracy is given by $v(\tau) \equiv (V - V_0)/V_0 \simeq \frac{1}{2} \epsilon^2 \delta^2 V/V_0$.

After a lengthy calculation and a similar consideration [17] as for δV , we obtain

$$\begin{aligned} \delta^2 V = & a_0^2 (1 - \cos \eta) \left(\mathcal{A}(\eta) \frac{\delta^2 E}{2} + \sin \eta \delta^2 C \right) \\ & + 12\pi a_0 \left(\mathcal{G}(\eta) J[\delta A, \delta A] + (1 + 2 \cos \eta) J[\delta \tau_0, \delta \tau_0] \right. \\ & \left. - 2(3 \sin \eta - \eta(1 + 2 \cos \eta)) J[\delta A, \delta \tau_0] \right), \end{aligned} \quad (27)$$

where $\delta^2 E$ and $\delta^2 C$ are the second variations of, respectively, the total energy E and the conserved quantity C :

$$\begin{aligned}\delta^2 E = & 24\pi \int_0^\pi \left(\delta^2 A + \frac{1}{3} \tan R \delta^2 A' \right) \sin^2 R \, dR \\ & + 24\pi \int_0^\pi \left(\frac{\delta A \delta f}{\cos^2 R} + \tan R \delta A \delta f' + \frac{\sin R}{\cos^3 R} \left(1 - \frac{2}{3} \sin^2 R \right) \delta f \delta A' \right) dR,\end{aligned}\quad (28)$$

$$\begin{aligned}\delta^2 C = & -12\pi \int_0^\pi \left(\delta^2 \tau_0 + \frac{1}{3} \tan R \delta^2 \tau_0' \right) \sin^2 R \, dR \\ & -12\pi \int_0^\pi \left(\frac{\delta \tau_0 \delta f}{\cos^2 R} + \tan R \delta \tau_0 \delta f' + \frac{\sin R}{\cos^3 R} \left(1 - \frac{2}{3} \sin^2 R \right) \delta f \delta \tau_0' \right) dR,\end{aligned}\quad (29)$$

and we have defined

$$J[\cdot, *] \equiv \int_0^\pi \left((\cdot)(*) + \frac{1}{3} \tan R ((\cdot)')(*) + (\cdot)(*)' \right) \sin^2 R \, dR, \quad (30)$$

and $\mathcal{G}(\eta) \equiv 7 - 8 \cos \eta + \cos^2 \eta - 6\eta \sin \eta + \eta^2(1 + 2 \cos \eta)$. (If we were allowed to apply “integrations by parts”, the above formulae would become much simpler, but because of the same reason as in the calculation of δV , we could not do so. A formal application would cause a divergence of the result.)

Since our variations are taken in the surface $P_{E,C}$ of constant E and C the variations of any order of E and C vanish, in particular, $\delta^2 E = 0$ and $\delta^2 C = 0$. Hence we obtain

$$\begin{aligned}\delta^2 V(\tau) = & 12\pi a_0 \left(\mathcal{G}(\eta) J[\delta A, \delta A] + (1 + 2 \cos \eta) J[\delta \tau_0, \delta \tau_0] \right. \\ & \left. - 2(3 \sin \eta - \eta(1 + 2 \cos \eta)) J[\delta A, \delta \tau_0] \right).\end{aligned}\quad (31)$$

This depends only upon the first variations of the functions $A(R)$ and $\tau_0(R)$.

We close this section by giving an explicit example. The three functions are $f(R) = \sin^2 R$, $A(R) = 1/3 + (1/900)(\sin 3R - 12 \sin 5R - 9 \sin 7R)$, and $\tau_0(R) = 0$. (This example has been made by taking $f(R) = \sin^2 R$ and putting

$$F(R) = \int_0^R f'(x) f^{\frac{1}{2}}(x) \psi(x) dx. \quad (32)$$

We can check that all the regularity conditions for $F(R)$ are satisfied if $\psi(R)$ is a positive function on $[0, \pi]$ such that it makes $F(\pi) = 0$. In particular, if we choose $\psi(R) = 1 + (1/10)((1/3) \sin 5R - \sin 7R)$, we have $F(R) = \sin^3 R(2/3 + (1/450)(\sin 3R - 12 \sin 5R - 9 \sin 7R))$, and the $A(R)$ presented above.) The corresponding FLRW solution as the averaged one is given by $a_0 = 1/3$ and $\tau_{0c} = 0$. FIGs.5 to 7 are, respectively, profiles of the metric components r and e^λ , and those of the energy density ε . FIG.8 shows the time-development of the total volume V . In each figure, the profiles

of the corresponding FLRW solution are also depicted with dashed curves. FIG.9 shows $\log_{10} \frac{\epsilon^2}{2} |\delta^2 V|/V_0$, which is the estimation of the accuracy $\log_{10} |v|$ by the second variation $\delta^2 V$, and the dashed curve shows the exact one $\log_{10} |V - V_0|/V_0$. The second variation $\delta^2 V$ is evaluated by putting $\epsilon \delta A = (1/900)(\sin 3R - 12 \sin 5R - 9 \sin 7R)$, so the integral $\epsilon^2 J[\delta A, \delta A] = -11\pi/324000 \simeq -10^{-4}$. We can see that the accuracy v is in this example better than 10^{-3} throughout the expansion phase, though the energy fluctuation becomes larger than 10^{-1} .

IV. CONCLUSIONS

We have seen that the FLRW solution is the critical point for the volume $V(\tau)$ in the space of spherically symmetric dust solutions on S^3 . In accordance with the fact that the FLRW solution contains two parameters (though one of which is redundant in a sense), we found that there is a natural foliation in the space of solutions defined by constant energy E and another quantity “ C ”. The exact statement of our result is that in each leaf of the foliation there exist a unique FLRW solution and this point is critical with respect to the variations taken in the same leaf. In our view, the “averaged” solution for all inhomogeneous solutions in a leaf corresponds to the FLRW solution in the same leaf.

Although our discussions have relied on the known exact solution for the spherically symmetric case, we have seen that the correspondence between Eqs.(20) and (24) is very natural and seems to be independent of the spherical symmetry. In fact, we have already obtained a preliminary result that supports a direct generalization of the present result [18]. This will appear elsewhere.

We could not discuss a relation to observables like the distance-redshift relation [19], which will be worth investigating further.

ACKNOWLEDGMENTS

The author acknowledges financial support from the Japan Society for the Promotion of Science and the Ministry of Education, Science and Culture.

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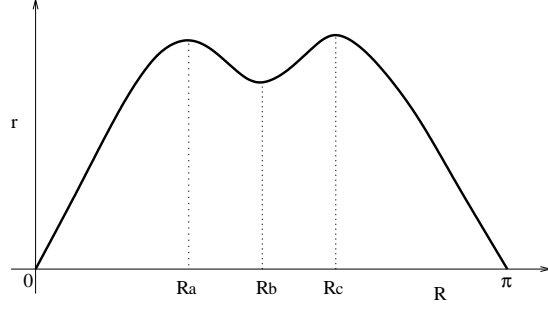


FIG. 1. A possible profile of r . There are three extremal points $R = R_a, R_b, R_c$ in this example.

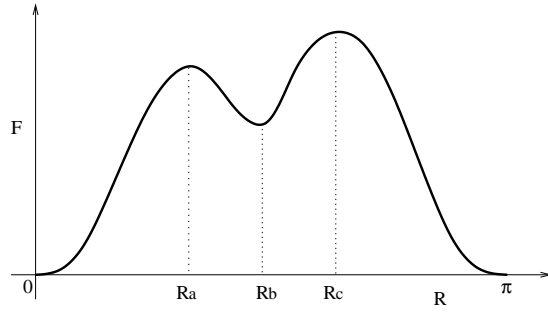


FIG. 2. A possible profile of F . The extremal points decide those for r .

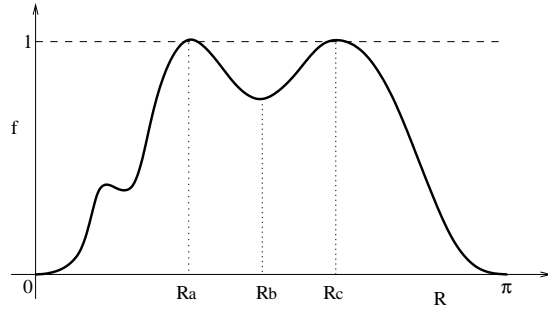


FIG. 3. A possible profile of f . At the extremal points of F , f should also be extremal, and at the maximal points among them f should take value 1.

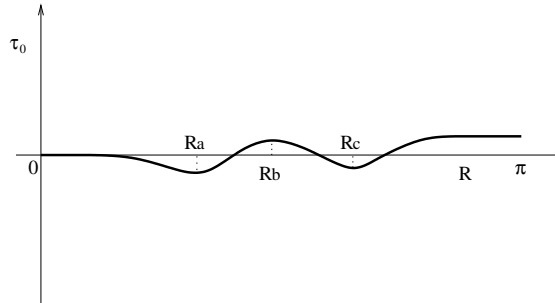


FIG. 4. A possible profile of τ_0 . The sign of the derivative should be opposite to that of F or equal to zero.

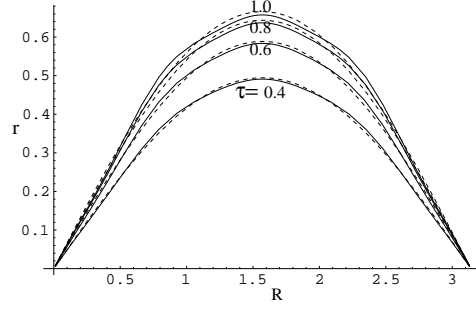


FIG. 5. Profiles of r for the example.

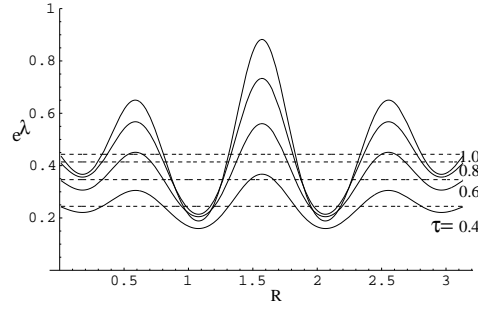


FIG. 6. Profiles of e^λ for the example.

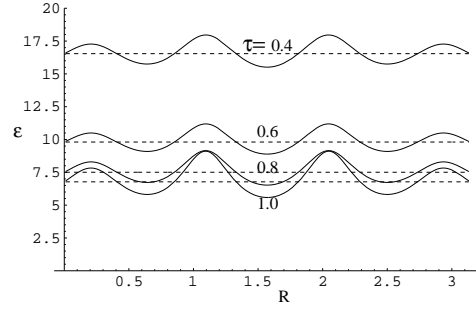


FIG. 7. Profiles of ε for the example.

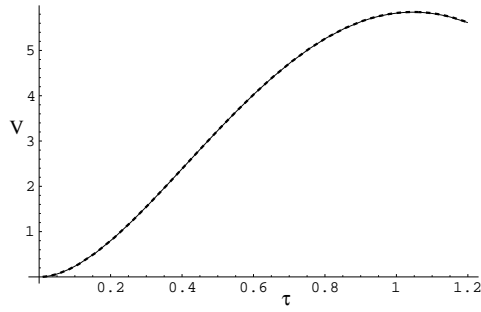


FIG. 8. The time-development of $V(\tau)$ for the example.

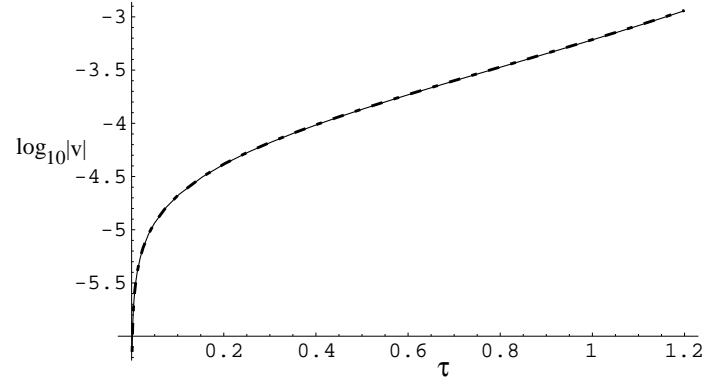


FIG. 9. Accuracy of $V_0(\tau)$ as the averaged solution — evaluated from the second variation $\log_{10} \frac{\epsilon^2}{2} |\delta^2 V| / 2V_0$ for the solid curve and from the exact one $\log_{10} |V - V_0| / V_0$ for the dashed curve.